

Representations of Leavitt Path Algebras

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Abstract

We study the (unital) representations of a Leavitt path algebra $L(\Gamma)$ of a finitely separated di(irected)graph Γ over a field. We show that the category of $L(\Gamma)$ -modules is equivalent to a subcategory of quiver representations. We give a necessary and sufficient criterion for the existence of a nonzero finite dimensional representation. When Γ is a (non-separated) row-finite digraph we determine all possible finite dimensional quotients of $L(\Gamma)$.

Keywords : Leavitt path algebra, quiver representation, finite dimensional module, nonstable K-theory, semigroup of a digraph, dimension function.

1 Introduction

Our aim is to understand and classify the representations (equivalently the module category) of the Leavitt path algebra $L(\Gamma)$ of a digraph Γ from the viewpoint of quiver representations. The main tool we employ is that the category of $L(\Gamma)$ -modules is equivalent to a subcategory of quiver representations of Γ satisfying a natural condition (Theorem 3.2). In particular, we give a necessary and sufficient condition for the existence of a (nonzero) finite dimensional representation (Corollary 3.7) and determine all finite dimensional quotients of $L(\Gamma)$ (Theorem 6.2). Before describing the contents of each section below we provide some context, background and motivation.

When R is an arbitrary ring with 1, two different bases of a finitely generated free R -module may not have the same number of elements. When they do, that is, if the free module R^m isomorphic to R^n as R -modules implies that $m = n$, we say that R has IBN (Invariant Basis Number). Commutative rings, division rings and Noetherian rings all have IBN.

A standard example of a ring without IBN is the endomorphism algebra of a countably infinite dimensional vector space, for instance $\mathbb{F}^{(\mathbb{N})}$, that is, functions

of finite support from \mathbb{N} to \mathbb{F} . If $R := \text{End } \mathbb{F}^{(\mathbb{N})}$ then R is isomorphic (as an R -module) to R^2 . Hence R^m is isomorphic to R^n for any two positive integers m and n . The isomorphisms between R and R^2 are given by $\alpha = \begin{bmatrix} D_0 \\ D_1 \end{bmatrix}$ in $M_{2 \times 1}(R)$ and $\beta = \begin{bmatrix} U_0 & U_1 \end{bmatrix}$ in $M_{1 \times 2}(R)$.

U_0 and U_1 (respectively, D_0 and D_1) are the upsampling (respectively, downsampling) operators of Signal Processing ([24], Chapter III Section 1):

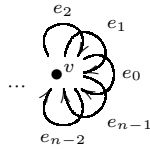
$$(U_0 f)(k) := \begin{cases} f(\frac{k}{2}) & k \text{ even} \\ 0 & k \text{ odd} \end{cases}, \quad (U_1 f)(k) := \begin{cases} 0 & k \text{ even} \\ f(\frac{k-1}{2}) & k \text{ odd} \end{cases};$$

$(D_0 f)(k) := f(2k)$, $(D_1 f)(k) := f(2k+1)$ for any f in $\mathbb{F}^{(\mathbb{N})}$. That $\alpha\beta = I_2$ and $\beta\alpha = I_1$ is easy to check.

Upsampling and downsampling by a factor of $n > 2$ is obtained by defining and $(U_i f)(k) = f(\frac{k-i}{n})$ if $k \equiv i \pmod n$ and 0 otherwise; $(D_i f)(k) = f(nk+i)$ for $0 \leq i < n$. Now $\alpha = \begin{bmatrix} D_0 & D_1 & \cdots & D_{n-1} \end{bmatrix}^T$ in $M_{n \times 1}(R)$ and $\beta = \begin{bmatrix} U_0 & U_1 & \cdots & U_{n-1} \end{bmatrix}$ in $M_{1 \times n}(R)$ satisfy $\alpha\beta = I_n$ and $\beta\alpha = I_1$. Thus the (unital) subalgebra A of $R = \text{End } \mathbb{F}^{(\mathbb{N})}$ generated by the upsampling and the downsampling operators $U_0, U_1, \dots, U_{n-1}, D_0, D_1, \dots, D_{n-1}$ satisfies $A \cong A^n$ but $A \not\cong A^j$ for $j = 2, \dots, n-1$. This algebra is a concrete realization of the algebra, nowadays called the Leavitt algebra and denoted by $L(1, n)$ defined (by generators and relations) and studied by Leavitt in [23].

When $\mathbb{F} = \mathbb{C}$ we can replace the vector space of finite \mathbb{C} -sequences $\mathbb{C}^{(\mathbb{N})}$ with the Hilbert space l^2 of square summable sequences. Then $\{U_i, D_i = U_i^*\}_{i=1}^n$ is a set of bounded linear operators (of norm 1) and the closure with respect to the operator norm of the $*$ -subalgebra of $B(l^2)$ generated by $\{U_i, U_i^*\}_{i=1}^n$ is the Cuntz algebra \mathcal{O}_n . In general, the Leavitt path algebra $L(\Gamma)$ is a dense $*$ -subalgebra of $C^*(\Gamma)$, the corresponding graph C^* -algebra [25, Theorem 7.3].

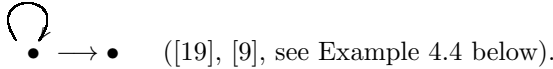
Leavitt defined $L(1, n)$ as the \mathbb{F} -algebra generated by $X_0, X_1, \dots, X_{n-1}, Y_0, Y_1, \dots, Y_{n-1}$ subject to the relations $Y_i X_j = \delta_{ij}$ for $0 \leq i, j < n$ and $X_0 Y_0 + X_1 Y_1 + \cdots + X_{n-1} Y_{n-1} = 1$. He proved that $L(1, n)$ is a simple algebra and $L(1, n) \cong L(1, n)^n$ but $L(1, n) \not\cong L(1, n)^j$ for $j = 2, \dots, n-1$, unlike $\text{End } \mathbb{F}^{(\mathbb{N})}$ above. Mapping X_i to U_i and Y_i to D_i defines an \mathbb{F} -algebra epimorphism from $L(1, n)$ to the algebra generated by the upsampling and downsampling operators. This is an isomorphism because $L(1, n)$ is simple. The algebra $L(1, n)$ is the Leavitt path algebra of R_n , the rose with n petals:



The Leavitt path algebra $L(\Gamma)$ of a di(irected)graph Γ was defined (many decades after Leavitt's seminal work, via a detour through functional analysis) by Abrams, Aranda Pino [2] and by Ara, Moreno, Pardo [12] (independently and essentially simultaneously) as an algebraic analog of a graph C^* -algebra. It is a Cohn localization of the path algebra $\mathbb{F}\Gamma$ of the digraph Γ [10, Corollary 4.2]. The excellent survey [1] is the definitive reference for the history and development of Leavitt path algebras. We will give the precise definition of $L(\Gamma)$ in the next section.

A major theme in the theory of Leavitt path algebras is to establish a dictionary between the graph theoretic properties of Γ and the algebraic structure of $L(\Gamma)$ (see [1] and the references within). In particular, in strictly ascending order of generality, for a finite digraph Γ it is known that:

- (i) Γ is acyclic (that is, Γ has no directed cycles) if and only if $L(\Gamma)$ has DCC (Descending Chain Condition) on right (or left) ideals [3, Theorem 2.6] if and only if $L(\Gamma)$ is von Neuman regular [7, Theorem 1] if and only if $L(\Gamma)$ is finite dimensional if and only if $L(\Gamma)$ is isomorphic to a direct sum of matrix algebras (over the ground field \mathbb{F}) [4, Corollaries 3.6 and 3.7].
- (ii) $L(\Gamma)$ has ACC (Ascending Chain Condition) on right (or left) ideals [3, Theorem 3.8] if and only if the cycles of Γ have no exits if and only if $L(\Gamma)$ is *locally finite dimensional* (i.e., a graded algebra with each homogeneous summand being finite dimensional) in which case $L(\Gamma)$ is isomorphic to a direct sum of matrix algebras over \mathbb{F} and/or matrix algebras over $\mathbb{F}[x, x^{-1}]$ (the Laurent polynomial algebra) [5, Theorems 3.8 and 3.10].
- (iii) The cycles in Γ are disjoint if and only if $L(\Gamma)$ has finite GK (Gelfand-Kirillov) dimension, that is, $L(\Gamma)$ has polynomial growth [8, Theorem 5]. In fact the previous items are special cases: Γ is acyclic if and only if the GK dimension of $L(\Gamma)$ is 0. The digraph Γ has a cycle but the cycles of Γ have no exits if and only if the GK dimension of $L(\Gamma)$ is 1. The first instance of $L(\Gamma)$ with GK dimension 2 is given by the Toeplitz digraph



We can add our Theorem 6.4 below to this list :

- (iv) $L(\Gamma)$ has a nonzero finite dimensional quotient if and only if Γ has a sink or a cycle which can not be reached from any other cycle.

Also our Corollary 6.5 states: If $L(\Gamma)$ has finite Gelfand-Kirillov dimension then $L(\Gamma)$ has a nonzero finite dimensional quotient and if $L(\Gamma)$ has a nonzero finite dimensional quotient then $L(\Gamma)$ has IBN. None of these implications are reversible.

Here is a summary of the contents of the rest of this paper: We review the relevant definitions and basic facts in the next section. In section 3, we work in the category $\mathfrak{M}_{L(\Gamma)}$ of unital modules over the Leavitt path algebra of a finitely separated digraph Γ . After observing that $\mathfrak{M}_{L(\Gamma)}$ is equivalent to a subcategory of the category of quiver representations of Γ (Theorem 3.2), we illustrate this point of view with several propositions and examples in sections 3 and 4. In particular, in Example 4.4 we give a short proof of the non-splitting theorem of [9, Theorem 2].

In section 3 we also give a necessary and sufficient criterion for the existence of a nonzero finite dimensional quotient in terms of dimension functions (Corollary 3.7). While this criterion is still difficult to check in the generality of separated digraphs, in the non-separated case we give an effective solution in terms of the digraph Γ .

In section 5 we collect a few definitions and facts needed in the last section. We also reinterpret the criterion for the existence of a nonzero finite dimensional representation in terms of the nonstable K -theory of $L(\Gamma) := L_{\mathbb{F}}(\Gamma, \Pi)$.

In section 6 we focus on non-separated digraphs. For these finitely separated is the same as row-finite. We determine all possible finite dimensional quotients of $L(\Gamma)$ for a row-finite digraph Γ : Any finite dimensional quotient of $L(\Gamma)$ is isomorphic to $\oplus M_{n_k}(B_k)$ where the sum is over maximal sinks and maximal cycles with finitely many predecessors in Γ and n_k is the number of paths in Γ terminating at the relevant sink or at a chosen vertex on the relevant cycle. The cyclic algebra B_k is $\mathbb{F}[x]/(P_k(x))$ with $P_k(0) = 1$. If k corresponds to a sink then $B_k = \mathbb{F}$ if this sink is in the support of M , $B_k = 0$ otherwise (Theorem 6.2).

2 Preliminaries

A *directed graph* Γ is a four-tuple (V, E, s, t) where V is the set of vertices, E is the set of arrows (directed edges), s and $t : E \rightarrow V$ are the source and the target functions. The digraph Γ is *finite* if E and V are both finite. Γ is *row-finite* if $s^{-1}(v)$ is finite for all v in V . Given $V' \subseteq V$ the *induced subgraph* on V' is $\Gamma' := (V', E', s', t')$ with $E' := s^{-1}(V') \cap t^{-1}(V')$; $s' := s|_{E'}$; $t' := t|_{E'}$. A subgraph is *full* if it is the induced subgraph on its vertices.

A vertex v in V is a *sink* if $s^{-1}(v) = \emptyset$; it is a *source* if $t^{-1}(v) = \emptyset$. An isolated vertex is both a source and a sink. For each e in E , $s(e)$ is the *initial vertex* of e and $t(e)$ is the *terminal vertex* of e . If $t(e) = s(e)$ then e is a *loop*. A path of length $n > 0$ is a sequence $p = e_1 \dots e_n$ such that $t(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. The source of p is $s(p) := s(e_1)$ and the target of p is $t(p) := t(e_n)$. A path p of length 0 consists of a single vertex v where $s(p) := v$ and $t(p) := v$. We will denote the length of p by $l(p)$. A path $C = e_1 e_2 \dots e_n$

with $n > 0$ is a *cycle* if $s(C) = t(C)$ and $s(e_i) \neq s(e_j)$ for $i \neq j$. An arrow $e \in E$ is an *exit* of the cycle $C = e_1 e_2 \cdots e_n$ if there is an i such that $s(e) = s(e_i)$ but $e \neq e_i$. The digraph Γ is *acyclic* if it has no cycles. An *infinite path* is an infinite sequence of arrows $e_1 e_2 e_3 \cdots$ such that $t(e_k) = s(e_{k+1})$ for $k = 1, 2, 3, \dots$.

Given a digraph Γ , the *extended digraph* of Γ is $\tilde{\Gamma} := (V, E \sqcup E^*, s, t)$ where $E^* := \{e^* \mid e \in E\}$ and the functions s and t are extended as $s(e^*) := t(e)$, $t(e^*) := s(e)$ for all $e \in E$. Thus the dual arrow e^* has the opposite orientation of e . We want to extend $*$ to an operator defined on all paths of $\tilde{\Gamma}$: Let $v^* := v$ for all v in V , $(e^*)^* := e$ for all e in E and $p^* := e_n^* \cdots e_1^*$ for a path $p = e_1 \cdots e_n$ with e_1, \dots, e_n in $E \sqcup E^*$. In particular $*$ is an involution, i.e., $** = id$.

A *separated digraph* is a pair (Γ, Π) where $\Gamma = (V, E, s, t)$ is a digraph and Π is a partition of E finer than $\{s^{-1}(v) : v \in V \text{ with } s^{-1}(v) \neq \emptyset\}$. That is, if e and f are in $X \in \Pi$ then $s(e) = s(f)$. Hence the induced source function $s : \Pi \rightarrow V$ is well-defined. We will also denote by X the function $E \rightarrow \Pi$ assigning to each arrow e the unique part $X \in \Pi$ containing e .

If $\Gamma' = (V', E')$ is a subgraph of the separated digraph Γ then Γ' is also a separated digraph with the separation $\Pi' := \{X \cap E' \mid X \in \Pi, X \cap E' \neq \emptyset\}$. A separated digraph is *finitely separated* if X is finite for all X in Π . Clearly, a subgraph of a finitely separated digraph is also finitely separated. For a non-separated digraph finitely separated is the same as row-finite, that is, $s^{-1}(v)$ is finite for every vertex v .

The *Leavitt path algebra of a separated digraph* (Γ, Π) with coefficients in the field \mathbb{F} , as defined in [11], is the \mathbb{F} -algebra $L_{\mathbb{F}}(\Gamma, \Pi)$ generated by $V \sqcup E \sqcup E^*$ satisfying:

- (V) $vw = \delta_{v,w} v$ for all $v, w \in V$,
- (E) $s(e)e = e = et(e)$ for all $e \in E \sqcup E^*$,
- (SCK1) $e^* f = \delta_{e,f} t(e)$ for all $e, f \in X$ and all $X \in \Pi$,
- (SCK2) $s(X) = \sum_{e \in X} ee^*$ for every finite $X \in \Pi$.

We will usually suppress the subscript \mathbb{F} when we denote our algebras. When Γ or Π are clear from the context we may also omit these from our notation.

The relations (V) simply state that the vertices are mutually orthogonal idempotents. If we only impose the relations (V) and (E) then we obtain $\mathbb{F}\tilde{\Gamma}$, the *path (or quiver) algebra* of the extended digraph $\tilde{\Gamma}$: The paths in $\tilde{\Gamma}$ form a vector space basis of $\mathbb{F}\tilde{\Gamma}$, the product pq of two paths p and q is their concatenation if $t(p) = s(q)$ and 0 otherwise. We get the *Cohn path algebra* $C(\Gamma, \Pi)$ of the separated digraph (Γ, Π) when we impose the relations (SCK1) in addition to (V) and (E). Hence $L(\Gamma, \Pi)$ is a quotient of $C(\Gamma, \Pi)$, which is a quotient of $\mathbb{F}\tilde{\Gamma}$. The abbreviation SCK stands for Separated Cuntz-Krieger.

Note that $L_{\mathbb{F}}(\Gamma)$ is not a quotient of the polynomial algebra in the noncommuting variables $V \sqcup E \sqcup E^*$ because we need to consider the nonunital algebra of polynomials without a constant term. In particular, when Γ is a single vertex v with no arrows then $L_{\mathbb{F}}(\Gamma) = \mathbb{F}v \cong \mathbb{F}$ not $\mathbb{F} + \mathbb{F}v$ (Similarly for the path algebra and also for the Cohn path algebra).

The algebras $\mathbb{F}\tilde{\Gamma}$, $C(\Gamma, \Pi)$ and $L(\Gamma, \Pi)$ are unital if and only if V is finite, in which case the sum of all the vertices is the unit: It is clear that $\sum_{v \in V} v = 1$ when V is finite. For the converse, a given element in any these algebras is a finite linear combination of paths in $\tilde{\Gamma}$ and we can pick $v \in V$ which is not the source of any of these paths if V is infinite. Now left multiplication by v gives zero, so there is no unit element in any of these algebras since Proposition 3.8 below shows that $v \neq 0$ in $L(\Gamma, \Pi)$ for every $v \in V$, hence also in $C(\Gamma, \Pi)$.

There is a \mathbb{Z} -grading on $\mathbb{F}\tilde{\Gamma}$ and all the other algebras above given by $|v| = 0$ for v in V , $|e| = 1$ and $|e^*| = -1$ for e in E . This defines a grading on all our algebras since all the relations are homogeneous. The linear extension of $*$ on paths induces a grade-reversing involutive anti-automorphism (i.e., $|\alpha^*| = -|\alpha|$ and $(\alpha\beta)^* = \beta^*\alpha^*$). Hence these algebras are \mathbb{Z} -graded $*$ -algebras and the (graded) categories of left modules and right modules for any of these algebras are equivalent.

More generally, we may consider G -gradings on $L(\Gamma, \Pi)$ and $C(\Gamma, \Pi)$ for any group G , with the generators $V \sqcup E \sqcup E^*$ being homogeneous. Since $v^2 = v$ and $e^*e = te$ we have: (i) $|v|_G = 1_G$ and (ii) $|e^*|_G = |e|_G^{-1}$. Conversely, any function from $V \sqcup E \sqcup E^*$ to G satisfying (i) and (ii) defines a G -grading on $L(\Gamma, \Pi)$ and $C(\Gamma, \Pi)$ as the remaining relations are homogeneous. A morphism (or a refinement) from a G -grading to an H -grading on the algebra A is given by a group homomorphism $\phi : G \rightarrow H$ such that for all $h \in H$, $A_h = \bigoplus_{\phi(g)=h} A_g$ where $A_g := \{a \in A : |a|_G = g\} \cup \{0\}$. There is a universal G -grading on $L(\Gamma, \Pi)$ and $C(\Gamma, \Pi)$ which is a refinement of all others:

Proposition 2.1 *Let $G := F_E$ be the free group on the set of arrows. The G -grading defined by $|v|_G = 1$, $|e|_G = e$ and $|e^*|_G = e^{-1}$ is an initial object in the category of G -gradings of $L(\Gamma, \Pi)$ or $C(\Gamma, \Pi)$ with the generators $V \sqcup E \sqcup E^*$ being homogeneous.*

Proof. For any H -grading let $\phi : G \rightarrow H$ be the homomorphism given by $\phi(e) = |e|_H$. ■

Combined with the existence of certain representations of $L(\Gamma, \Pi)$ defined in the next section this universal grading is useful in showing that some elements of $L(\Gamma, \Pi)$ are nonzero (or linearly independent) as in Proposition 3.8 below.

When $\Pi = \{s^{-1}(v) \mid v \in V, s^{-1}(v) \neq \emptyset\}$, we say that Γ is not separated, $C(\Gamma, \Pi)$ is denoted by $C(\Gamma)$ and called the *Cohn path algebra* of Γ . Similarly

$L(\Gamma, \Pi)$ is denoted by $L(\Gamma)$ and called the *Leavitt path algebra* of Γ . Also the conditions (SCK1) and (SCK2) are denoted by (CK1) and (CK2) respectively [2], [12].

For any arrow e in E we have $e^*e = t(e)$ by (SCK1). Consequently $p^*p = t(p)$ for any path p of Γ . Hence for any two paths p and q of Γ if $q = pr$ then $p^*q = p^*pr = r$, if $p = qr$ then $p^*q = (q^*p)^* = r^*$. If Γ is not separated then (CK1) also implies that $e^*f = 0$ when $e \neq f$ in E . Hence $p^*q = 0$ unless the path q is an initial segment of the path p ($p = qr$) or p is an initial segment of q ($q = pr$). Thus the Cohn path algebra $C(\Gamma)$ and the Leavitt path algebra $L(\Gamma)$ are spanned by $\{pq^*\}$ where p and q are paths of Γ with $t(p) = t(q)$. In fact this is a basis for $C(\Gamma)$ which can be shown by defining an epimorphism from $C(\Gamma)$ to a reduced semigroup algebra with this basis (we will not need this fact). In $L(\Gamma)$ however $\{pq^* : t(p) = t(q)\}$ is linearly dependent because of (CK2).

A subset H of V is *hereditary* if for any path p , $s(p) \in H$ implies that $t(p) \in H$ [2]; H is Π -*saturated* if $\{t(e) : e \in X\} \subseteq H$ for some (finite) $X \in \Pi$ implies that $s(X) \in H$ [11]. If I is an ideal of $L(\Gamma, \Pi)$ and p is a path in Γ with $s(p) \in I$ then $t(p) = p^*p = p^*s(p)p \in I$, also if $\{t(e) : e \in X\} \subseteq I$ then $s(X) = \sum_{e \in X} ee^* = \sum_{e \in X} et(e)e^* \in I$, so $I \cap V$ is hereditary and Π -saturated. We have a Galois correspondence between the subsets of V and the ideals of $L(\Gamma, \Pi)$ given by $S \mapsto (S)$ and $I \mapsto I \cap V$ which gives a bijection between hereditary saturated subsets of V and graded ideals of $L(\Gamma)$ when Γ is a (non-separated) row-finite digraph [12, Theorem 5.3].

3 Quiver Representations and $L(\Gamma, \Pi)$ -Modules

We will work in the category \mathfrak{M}_L of unital (right) modules over $L := L_{\mathbb{F}}(\Gamma, \Pi)$. However L has a 1 if and only if the vertex set V is finite. Even if V is infinite, we define a unital L -module as a module M with the property that $ML = M$, i.e., for any m in M we can find $\lambda_1, \lambda_2, \dots, \lambda_n$ in L and m_1, m_2, \dots, m_n in M so that $m = m_1\lambda_1 + m_2\lambda_2 + \dots + m_n\lambda_n$. This condition is equivalent to the standard definition of unital (when L has a 1) since $m1 = (m_1\lambda_1 + m_2\lambda_2 + \dots + m_n\lambda_n)1 = m_1\lambda_11 + m_2\lambda_21 + \dots + m_n\lambda_n1 = m_1\lambda_1 + m_2\lambda_2 + \dots + m_n\lambda_n = m$. The category of unital modules is an abelian category with sums since it is closed under taking quotients, submodules, extensions, (arbitrary) sums (but not infinite products: if V is infinite then the L -module L^V is not unital).

From now on we will omit the parentheses to reduce notational clutter when the source and target functions s, t are applied. We will need the following consequence of (SCK1) and (SCK2):

Lemma 3.1 *Let Γ be a finitely separated digraph and M be a right L -module. Then $(\mu_e)_{e \in X} : MsX \longrightarrow \oplus_{e \in X} Mte$ for all $X \in \Pi$ is an isomorphism (of vector spaces), where μ_e is right multiplication by e . Moreover, if M is unital then $M = \oplus_{v \in V} Mv$ (as a vector space) and $MsX = \oplus_{e \in X} Mee^*$.*

Proof. To see that the inverse of $(\mu_e)_{e \in X}$ is $[\mu_{e^*}]_{e \in X} : \oplus_{e \in X} Mte \longrightarrow MsX$ given by $(m_e)_{e \in X} [\mu_{e^*}]_{e \in X} = \sum_{e \in X} m_e e^*$ we check their compositions:

$$(m_e)_{e \in X} [\mu_{e^*}]_{e \in X} (\mu_f)_{f \in X} = \left(\sum_{e \in X} m_e e^* \right) (\mu_f)_{f \in X} = \left(\sum_{e \in X} m_e e^* f \right)_{f \in X} = (m_f)_{f \in X}$$

where the last equality uses the relation $e^* f = \delta_{e,f} tf$ and $m_f(tf) = m_f$ since $m_f \in Mtf$. Also for $m \in MsX$ we get

$$m(\mu_e)_{e \in X} [\mu_{e^*}]_{e \in X} = (me)_{e \in X} [\mu_{e^*}]_{e \in X} = \sum_{e \in X} mee^* = msX = m.$$

When M is unital for any m in M we have $m = \sum_{k=1}^n m_k \alpha_k = \sum_{j=1}^l m'_j v_j$ for some vertices $v_1, \dots, v_l \in V$. Hence $M = \sum_{v \in V} Mv$. This sum is direct: For any finite set Λ of vertices with $v \notin \Lambda$ if $m \in Mv \cap \sum_{w \in \Lambda} Mw$ then $m = mv$ (since $m \in Mv$) but $mv = 0$ (since $vw = 0$ for each $w \in \Lambda$). Thus $Mv \cap \sum_{w \in \Lambda} Mw = 0$.

$MsX = \sum_{e \in X} Mee^*$ since $sX = \sum_{e \in X} ee^*$. If $e \neq f$ in X then $ee^* ff^* = e(e^* f) f^* = 0$. Also $ee^* ee^* = ee^*$. That is, $MsX = \oplus_{e \in X} Mee^*$. ■

Consequently, the linear transformation defined by right multiplication with any e in E from Mse to Mte is onto. Hence right multiplication with any path p from Msp to Mtp is also onto. Similarly, right multiplication with p^* is injective.

We want to view the category \mathfrak{M}_L as a subcategory of $\mathfrak{M}_{\mathbb{F}\Gamma}$, the category of unital modules over the path algebra $\mathbb{F}\Gamma$, or equivalently, the category of quiver representations of Γ . The category of quiver representations of Γ is the category of functors from the path category of the digraph Γ (whose objects are the vertices V and the morphisms are the paths in Γ) to the category of \mathbb{F} -vector spaces. A morphism of quiver representations is a natural transformation between two such functors. That is, a quiver representation ρ assigns a (possibly infinite dimensional) vector space $\rho(v)$ to each vertex v and a linear transformation $\rho(e) : \rho(se) \longrightarrow \rho(te)$ to each arrow e . A morphism of quiver representations $\varphi : \rho \longrightarrow \sigma$ is a family of linear transformations $\{\varphi_v : \rho(v) \longrightarrow \sigma(v)\}_{v \in V}$ such

$$\text{that } \forall e \in E \text{ the diagram } \begin{array}{ccc} \rho(se) & \xrightarrow{\rho(e)} & \rho(te) \\ \varphi_{se} \downarrow & & \downarrow \varphi_{te} \\ \sigma(se) & \xrightarrow{\sigma(e)} & \sigma(te) \end{array} \text{ commutes [17].}$$

In Theorem 3.2 below the hypothesis on Γ of being finitely separated may be removed (even in the generality of Cohn-Leavitt path algebras of separated digraphs as defined in [11]) at the cost of complicating condition (I). We will not pursue this generality here.

Theorem 3.2 *If $\Gamma = (V, E, s, t, \Pi)$ is a finitely separated digraph then the category \mathfrak{M}_L is equivalent to the category of quiver representations ρ of Γ satisfying:*

$$\text{For all } X \in \Pi, \quad (\rho(e))_{e \in X} : \rho(sX) \longrightarrow \bigoplus_{e \in X} \rho(te) \text{ is an isomorphism.} \quad (I)$$

Proof. Given a right L -module M we define a quiver representation ρ_M as follows: $\rho_M(v) = Mv$ and $\rho_M(e) : Mse \longrightarrow Mte$ is defined by $m(se)\rho_M(e) := m(se)e = me(te)$. By the first part of Lemma 3.1 (I) is satisfied. If $\varphi : M \longrightarrow N$ is an L -module homomorphism then φ_v is the linear transformation making the

$$\begin{array}{ccc} \rho_M(v) = Mv & \hookrightarrow & M \\ \varphi_v \downarrow & & \downarrow \varphi \\ \rho_N(v) = Nv & \hookrightarrow & N \end{array} \text{ commutative. This defines a homomorphism}$$

of quiver representations, i.e., $\rho_M(e)\varphi_{te} = \varphi_{se}\rho_N(e)$ because right multiplication by e commutes with φ .

Given a quiver representation ρ we define (underlying vector space of) the corresponding module $M_\rho = \bigoplus_{v \in V} \rho(v)$. To define L -module structure we will use the projections $p_v : \bigoplus_{w \in V} \rho(w) \longrightarrow \rho(v)$, the inclusions $\iota_v : \rho(v) \longrightarrow \bigoplus_{w \in V} \rho(w)$ for $v \in V$ and the projections $p_e : \bigoplus_{f \in X_e} \rho(tf) \longrightarrow \rho(te)$, the inclusions $\iota_e : \rho(te) \hookrightarrow \bigoplus_{f \in X_e} \rho(tf)$. Now, let $mv := mp_v \iota_v$, $me := mp_{se} \rho(e) \iota_{te}$, $me^* := mp_{te} \iota_e (\rho(f))_{f \in X_e}^{-1} \iota_{sX}$. It is routine (albeit tedious) to check that the defining relations of L are satisfied.

Now let's check that the constructions above yield equivalences of categories. $M_{\rho_M} := \bigoplus_{v \in V} Mv = M$ by Lemma 3.1, as vector spaces. It is easy to check that the L -module structures also match. Also, given a module homomorphism $\varphi : M \longrightarrow N$, we have $\varphi = \bigoplus_{v \in V} \varphi_v : \bigoplus_{v \in V} Mv \longrightarrow \bigoplus_{v \in V} Nv$. For the composition in the other order $\rho_{M_\rho}(v) := M_\rho v := (\bigoplus_{w \in V} \rho(w))v = \rho(v)$ and $\rho(e) = \rho_{M_\rho}(e) : M_\rho se \longrightarrow M_\rho te$ because the diagram

$$\begin{array}{ccc} M_\rho se = \rho(se) & \xrightarrow{\iota_{se}} & M_\rho := \bigoplus_{w \in V} \rho(w) \\ \rho(e) \downarrow & & \downarrow p_{se} \rho(e) \iota_{te} \\ M_\rho te = \rho(te) & \xrightarrow{\iota_{te}} & M_\rho := \bigoplus_{w \in V} \rho(w) \end{array}$$

commutes. Finally, for any homomorphism $\{\varphi_v : \rho(v) \longrightarrow \sigma(v)\}_{v \in V}$ from ρ to σ , the v -component of $\bigoplus_{w \in V} \varphi_w$ is $\varphi_v : \rho_{M_\rho}(v) = \rho(v) \longrightarrow \sigma_{M_\sigma}(v) = \sigma(v)$. ■

The argument above almost proves the stronger statement that \mathfrak{M}_L is *isomorphic* to the subcategory of quiver representations of Γ satisfying the condition (I). The only issue is the difference between internal and external direct sums. In fact, we can obtain an isomorphism of categories if we work in a graded category where each subspace Mv of the L -module M , $v \in V$, is a homogeneous summand. There is no need for such an artifice since equivalence of categories is sufficient for our purposes.

$L = \oplus vL$ where the sum is over $v \in V$ and each vL is a cyclic projective L -module. The vector space $Mv \cong \text{Hom}^L(vL, M)$ is actually a module over the corner algebra $vLv \cong \text{End}^L(vL)$.

\mathfrak{M}_L is a categorical localization of the quiver representations of Γ . This is related to the fact that the algebra L is a Cohn localization of the path algebra $\mathbb{F}\Gamma$, when Γ is non-separated this is Corollary 4.2 of [10]. Recall that the Cohn localization $\Sigma^{-1}A$ of an algebra A with respect to a set $\Sigma = \{\sigma : P_\sigma \rightarrow Q_\sigma\}$ of homomorphisms between finitely generated projective A -modules, is an initial object among algebra homomorphisms $f : A \rightarrow B$ such that $\sigma \otimes id : P_\sigma \otimes_A B \rightarrow Q_\sigma \otimes_A B$ is an isomorphism for every σ in Σ .

Proposition 3.3 *If Γ is a finitely separated digraph then $L(\Gamma, \Pi)$ is the Cohn localization of $\mathbb{F}\Gamma$ with respect to $\{\sigma_X : \oplus_{e \in X} (te)\mathbb{F}\Gamma \rightarrow (sX)\mathbb{F}\Gamma \mid X \in \Pi\}$ where $\sigma_X((a_e)_{e \in X}) := \sum_{e \in X} ea_e$.*

Proof. For any $v \in V$ the cyclic (right) module $v\mathbb{F}\Gamma$ is projective since v is an idempotent. For all X in Π , $\sigma_X \otimes id_{L(\Gamma, \Pi)}$ is an isomorphism with inverse $(e^* \cdot)_{e \in X}$ where $e^* \cdot$ denotes left multiplication by e^* . When $f : \mathbb{F}\Gamma \rightarrow B$ is an algebra homomorphism $f(v)^2 = f(v)$ and $v\mathbb{F}\Gamma \otimes_{\mathbb{F}\Gamma} B \cong f(v)B$ via $a \otimes b \mapsto f(a)b$ and $b \mapsto v \otimes b$ (note that $\mathbb{F}\Gamma$ and B need not be unital). If $f : \mathbb{F}\Gamma \rightarrow B$ is an algebra homomorphism such that $\sigma_X \otimes id_B$ is an isomorphism for all X in Π then the composition $f(sX)B \cong (sX)\mathbb{F}\Gamma \otimes_{\mathbb{F}\Gamma} B \xrightarrow{\sigma_X^{-1}} \left(\oplus (te')\mathbb{F}\Gamma \right) \otimes_{\mathbb{F}\Gamma} B \cong \oplus f(te')B \xrightarrow{pr_e} f(te)B$ is uniquely and completely determined by the image of $f(sX)$, which we call $f(e^*)$. Now $\tilde{f}(v) := f(v)$ for all v in V , $\tilde{f}(e) := f(e)$ for all e in $E \sqcup E^*$ defines the unique homomorphism $\tilde{f} : L(\Gamma, \Pi) \rightarrow B$ factoring f through $\mathbb{F}\Gamma \rightarrow L(\Gamma, \Pi)$. ■

With the quiver representation viewpoint there is no need to mention the generators $\{e^* : e \in E\}$ explicitly, they are implicit in the condition (I). Theorem 3.2 also enables us to construct concrete models for L -modules and homomorphisms between them as illustrated in the following applications.

Proposition 3.4 *Let Γ be a finitely separated digraph. If $d : V \rightarrow \mathbb{N} \cup \{\infty\}$ satisfies $d(sX) = \sum_{e \in X} d(te)$ for all X in Π then there is an L -module M with $\dim^{\mathbb{F}}(Mv) = d(v)$.*

Proof. Let the quiver representation ρ be given by $\rho(v) := \mathbb{F}^{d(v)}$ if $d(v) < \infty$ and $\rho(v) := \mathbb{F}^{(\mathbb{N})}$ otherwise. We can find isomorphisms $\theta_X : \rho(sX) \rightarrow \oplus_{e \in X} \rho(te)$ for all X in Π by the hypothesis on d . Let $\rho(e) := \theta_X pr_e$ for all e in E . Condition (I) is satisfied by construction and the corresponding L -module M of Theorem 3.2 has $\dim^{\mathbb{F}}(Mv) = \dim^{\mathbb{F}} \rho(v) = d(v)$. ■

Corollary 3.5 *There is an L -module M with $Mv \cong \mathbb{F}^{(\mathbb{N})}$ for all v in V . Hence p and p^* are nonzero in L for every path p of Γ .*

Proof. The existence of an L -module M with $Mv \cong \mathbb{F}^{(\mathbb{N})}$ for all v in V is given by Proposition 3.4. Right multiplication by p is onto from Msp to $Mtp \cong \mathbb{F}^{(\mathbb{N})}$ by Lemma 3.1. Hence every path p in L is nonzero. Since $*$ is an involution p^* also is nonzero. ■

Definition 3.6 A *dimension function* of a finitely separated digraph Γ is a function $d : V \longrightarrow \mathbb{N}$ satisfying: $d(sX) = \sum_{e \in X} d(te)$ for all X in Π .

If the L -module M is *finitary*, that is, $\dim(Mv) < \infty$ for all v in V then Lemma 3.1 shows that $d(v) := \dim(Mv)$ is a dimension function. The converse also holds, that is, every dimension function is realizable:

Corollary 3.7 If d is a dimension function for Γ then there exists an L -module M with $\dim(Mv) = d(v)$. Hence L has a nonzero finite dimensional module if and only if Γ has a nonzero dimension function of finite support.

Proof. This is the special case of Proposition 3.4 with $d : V \longrightarrow \mathbb{N}$. By Lemma 3.1, $\dim(M) = \sum_{v \in V} \dim(Mv)$, hence $d(v) = \dim(Mv)$ has finite support if M is finite dimensional. ■

If M is a nonzero finite dimensional L -module then the image of L in $\text{End}^{\mathbb{F}} M$ is (isomorphic to) a nonzero finite dimensional quotient of L . Conversely a nonzero finite dimensional quotient is also a (nonzero finite dimensional) L -module. Thus the corollary above gives a necessary and sufficient condition for the existence of nonzero finite dimensional quotient. When Γ is a (non-separated) row-finite digraph we determine all possible finite dimensional quotients of $L(\Gamma)$ in Theorem 6.2 of Section 6.

The next proposition shows that the natural algebra homomorphism from $\mathbb{F}\Gamma$ to $L(\Gamma, \Pi)$ is injective. (This is still true when Γ is not finitely separated.) Thus we may regard the path algebra $\mathbb{F}\Gamma$ as a subalgebra of the Leavitt path algebra L . For a non-separated digraph this is Lemma 1.6 of [18].

Proposition 3.8 If Γ is a finitely separated digraph then the homomorphism from the path algebra $\mathbb{F}\Gamma$ to the Leavitt path algebra L is injective.

Proof. Let F_E be the free group on E (the arrow set of Γ). We can define F_E -gradings on $\mathbb{F}\Gamma$ and L by $|v| = 1$ for all $v \in V$, $|e| = e$ and $|e^*| = e^{-1}$ for all $e \in E$ (since all the relations are homogeneous this grading is well-defined). The homomorphism $\mathbb{F}\Gamma \longrightarrow L$ is graded so its kernel is a graded ideal. Any homogeneous element of $\mathbb{F}\Gamma$ is either a scalar multiple of some path p of positive length or a linear combination of vertices. By Proposition 3.4 there is an L -module M with $\dim(Mv) = \infty$ for every v in V . Vertices of Γ are orthogonal idempotents of L defining projections on M with infinite dimensional images, so a linear combination of vertices will be zero in L if and only if it is trivial. Also, Lemma 3.1 implies that the linear transformation given by right multiplication with p of positive length from Msp to Mtp is onto, hence $p \neq 0$ in L . Therefore

the kernel of $\mathbb{F}\Gamma \rightarrow L$ is trivial. ■

Lemma 1.6 of [18] actually states that the set of all paths and all dual paths $\{p\} \cup \{p^*\}$ is linearly independent in L for a non-separated digraph Γ . The proof above yields this stronger statement for a finitely separated Γ since all elements of $\{p\} \cup \{p^*\}$ are nonzero (by Corollary 3.5) homogeneous and they have different grades.

4 Examples of Representations for Non-separated Digraphs

Example 4.1 When Γ is the rose with n petals R_n , let $M = \rho(v) := \mathbb{F}^{(\mathbb{N})}$ and let $\rho(e_i) : \rho(v) \rightarrow \rho(v)$ be given by $(a_0 a_1 a_2 \cdots) \rho(e_i) = (a_i a_{i+n} a_{i+2n} \cdots)$ where e_i , $i = 0, \dots, n-1$ are the loops of Γ . Condition (I) is satisfied, $\rho(e_i)$ are the downsampling maps and this representation of $L(R_n)$ in $\text{End } \mathbb{F}^{(\mathbb{N})}$ gives the realization of $L(1, n)$ mentioned in the introduction.

Example 4.2 Let Γ be a row-finite digraph, w a sink in Γ and P^w the set of all paths ending at w . We will define the $L(\Gamma)$ -module M^w via the corresponding quiver representation ρ^w where $\rho^w(v)$ is the vector space with basis $\{p \in P^w \mid sp = v\}$ and $\rho^w(e)$ is the linear transformation defined as

$$p\rho^w(e) := \begin{cases} w & \text{if } p = e \\ e_2 e_3 \cdots e_n & \text{if } p = ee_2 e_3 \cdots e_n \\ 0 & \text{otherwise.} \end{cases}$$

Grouping the paths from v to w by their first arrow gives Condition (I) since for every nonsink v there is a bijection between the disjoint union of the given bases of $\rho^w(te)$ over $e \in s^{-1}(v)$ and the given basis of $\rho^w(v)$.

P^w is an \mathbb{F} -basis of M^w and (the proof of Theorem 3.2 shows that) $pe^* = ep$. Hence the image of pq^* in $\text{End}^{\mathbb{F}}(M^w)$ with p, q in P^w is the elementary matrix E_{pq} , thus M^w is simple. The two-sided ideal (w) of $L(\Gamma)$ is spanned by $\{pq^* \mid p, q \text{ in } P^w\}$ which is linearly independent in $L(\Gamma)$ since the image set $\{E_{pq}\}$ is linearly independent. Therefore $(w) \cong M_{n(w)}(\mathbb{F})$, the algebra of matrices indexed by P^w with only finitely many nonzero entries where $n(w)$ is the number of paths ending at w .

Mapping $p \in P^w$ to p^* defines a homomorphism from M^w to $wL(\Gamma)$ which is onto: $\{pq^* \mid sp = w, tp = tq\}$ spans $wL(\Gamma)$, but w is a sink so $p = w = tq$ and $\{q^* \mid tq = w\}$ spans $wL(\Gamma)$. Since M^w is simple and $wL(\Gamma)$ is nonzero by Proposition 3.8, $M^w \cong wL(\Gamma)$ thus M^w is projective. If N is a finite direct sum of $\{M^w\}$ then $\dim^{\mathbb{F}}(Nu)$ is the multiplicity of M^u for any sink u . Hence there are no relations among the isomorphism classes of distinct M^w . In particular if $u \neq w$ are sinks then $M^u \not\cong M^w$.

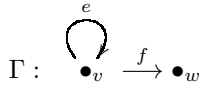
Defining the grade of p in P^w to be $-l(p)$ makes M^w a graded $L(\Gamma)$ -module. Then E_{pq} is a graded homogeneous linear transformation of degree $l(p) - l(q)$. If for every vertex v in Γ there is a path from v to a sink then we have a monomorphism from $L(\Gamma)$ to $\oplus \text{End}^{\mathbb{F}}(M^w)$ where the sum is over all sinks of Γ (because this is a graded homomorphism whose kernel does not contain any vertex).

When Γ is finite and acyclic then $\text{End}^{\mathbb{F}}(M^w) \cong M_{n(w)}(\mathbb{F})$ and the homomorphism from $L(\Gamma)$ to $\oplus M_{n(w)}(\mathbb{F})$ is onto since all the elementary matrices are in its image. Acyclicity of Γ and (CK2) yields that $\{pq^* \mid tp = tq = \text{sink}\}$ spans $L(\Gamma)$. Their images $\{E_{pq}\}$ are linearly independent, so $L(\Gamma) \cong \oplus M_{n(w)}(\mathbb{F})$. Thus M^w are the only simple modules of $L(\Gamma)$ and also $L(\Gamma)$ is finite dimensional. Conversely, if $L(\Gamma)$ is finite dimensional then Γ has finitely many vertices and arrows as these are part of a basis of $\mathbb{F}\Gamma \subseteq L(\Gamma)$. If Γ had a cycle C then C^k for $k = 1, 2, \dots$ would be linearly independent in $\mathbb{F}\Gamma \subseteq L(\Gamma)$ contradicting fact that $L(\Gamma)$ is finite dimensional. Hence $L(\Gamma)$ is finite dimensional if and only if Γ is finite and acyclic [4, Corollary 3.6].

The discussion above applies verbatim, proving a generalization to infinite digraphs:

Proposition 4.3 *If Γ is row-finite and has no infinite paths then $L(\Gamma) \cong \oplus M_{n(w)}(\mathbb{F})$ where the sum is over all sinks. (Here $M_{n(w)}(\mathbb{F})$ is the algebra of matrices indexed by P^w with only finitely many nonzero entries.) Also $M^w \cong wL(\Gamma)$ is a graded simple projective module for every sink w .*

Example 4.4 *An important instance of Example 4.2 is the Toeplitz digraph:*



The basis given above of M^w is $\{w, f, ef, e^2f, \dots\}$ which can be identified with \mathbb{N} via the length function. Therefore $M^w \cong \mathbb{F}^{(\mathbb{N})}$ as vector spaces where $a_0w + a_1f + a_2ef + \dots$ corresponds to $(a_0 \ a_1 \ a_2 \ \dots)$, a finite \mathbb{F} -sequence. This representation of $L(\Gamma)$ also fits the framework of Proposition 3.4 with $d(v) = \infty$ and $d(w) = \dim M^w w = \dim wL(\Gamma)w = 1$.

If S and T denote the images of $e + f$ and $e^* + f^*$ in $\text{End } \mathbb{F}^{(\mathbb{N})}$, respectively then $(a_0 \ a_1 \ a_2 \ \dots)S = (a_1 \ a_2 \ a_3 \ \dots)$ and $(a_0 \ a_1 \ a_2 \ \dots)T = (0 \ a_0 \ a_1 \ a_2 \ \dots)$. We have: $v + w = 1$, $(e + f)(e^* + f^*) = ee^* + ff^* = v$, $(e + f)w = f$, $w(e^* + f^*) = f^*$ showing that $e + f$ and $e^* + f^*$ generate $L(\Gamma)$.

Since $(e^* + f^*)(e + f) = e^*e + f^*f = v + w = 1$, we have an epimorphism from the Jacobson [19] algebra $\mathbb{F}\langle x, y \rangle := \mathbb{F}\langle X, Y \rangle / (1 - YX)$ to $L(\Gamma)$ sending x to $e + f$ and y to $e^* + f^*$. Composing this with the homomorphism from $L(\Gamma)$ to $\text{End } \mathbb{F}^{(\mathbb{N})}$ gives a monomorphism as $\{x^m y^n \mid m, n \in \mathbb{N}\}$ spans the Jacobson algebra and their images $\{S^m T^n \mid m, n \in \mathbb{N}\}$ are linearly independent. Thus $L(\Gamma)$ is isomorphic to the Jacobson algebra and also the subalgebra of $\text{End } \mathbb{F}^{(\mathbb{N})}$

generated by S and T .

$L(\Gamma)/(w) \cong \mathbb{F}[x, x^{-1}]$ since $w \leftrightarrow 1 - xy$ in the isomorphism between $L(\Gamma)$ and $\mathbb{F}\langle x, y \rangle$ above. The short exact sequence $M_\infty(\mathbb{F}) \cong (w) \hookrightarrow L(\Gamma) \twoheadrightarrow \mathbb{F}[x, x^{-1}]$ does not split [9, Theorem 2]: If it were split then there would be a subalgebra A of $\text{End}\mathbb{F}^{(\mathbb{N})}$ generated by $S + \alpha$ and $T + \beta$ isomorphic to $\mathbb{F}[x, x^{-1}]$ with $x \leftrightarrow S + \alpha$ and $x^{-1} \leftrightarrow T + \beta$, for some α and β with finite dimensional images. Considering $\mathbb{F}^{(\mathbb{N})}$ as a right $A \cong \mathbb{F}[x, x^{-1}]$ -module we see that $S + \alpha$ and $T + \beta$ are inverses of each other. There is a k with $\mathbb{F}^{(\mathbb{N})}\alpha \subseteq \mathbb{F}^k := \{a_0a_1 \cdots \mid a_n = 0 \text{ for } n \geq k\}$ and so $\mathbb{F}^{k+1}(S + \alpha) \subseteq \mathbb{F}^k$ because $\mathbb{F}^{k+1}S = \mathbb{F}^k$. Thus $S + \alpha$ has a nontrivial kernel, contradicting that $S + \alpha$ is invertible.

The short exact sequence $M_\infty(\mathbb{F}) \cong (w) \hookrightarrow L(\Gamma) \twoheadrightarrow \mathbb{F}[x, x^{-1}]$ does not split as $L(\Gamma)$ -modules either since $M_\infty(\mathbb{F})$ is not finitely generated. However, the inclusion $wL(\Gamma) \hookrightarrow L(\Gamma)$ does split since $v + w = 1$, hence $vL(\Gamma) \oplus wL(\Gamma) = L(\Gamma)$. By Proposition 3.3, $vL(\Gamma) \cong vL(\Gamma) \oplus wL(\Gamma)$ thus $L(\Gamma) \cong wL(\Gamma) \oplus L(\Gamma) \cong (wL(\Gamma))^n \oplus L(\Gamma)$ for any $n \in \mathbb{N}$. Therefore the category of finitely generated $L(\Gamma)$ -modules does not have Krull-Schmidt because $wL(\Gamma)$ is simple by Example 4.2, hence indecomposable. (More generally, if Γ is a finite digraph containing a cycle and a path from this cycle to a sink then the category of finitely generated representations of $L(\Gamma)$ does not have Krull-Schmidt.)

When $\mathbb{F} = \mathbb{C}$ we can replace the vector space of finite \mathbb{C} -sequences $\mathbb{C}^{(\mathbb{N})}$ with the Hilbert space l^2 of square summable sequences. Then S and $T = S^*$ above are bounded operators (of norm 1) and the closure with respect to the operator norm of the $*$ -subalgebra $L(\Gamma)$ in the C^* -algebra of bounded linear operators $B(l^2)$ generated by S and S^* is the classical Toeplitz algebra.

Example 4.5 (*Chen modules [16]*) Let Γ be a row-finite digraph, $\alpha = e_1e_2e_3 \cdots$ an infinite path and $[\alpha]$ the set of infinite paths $\beta = f_1f_2f_3 \cdots$ having the same tail as α (that is, $f_{m+k} = e_{n+k}$ for all k in \mathbb{N} , for some m and n). We will define the $L(\Gamma)$ -module M^α via the quiver representation ρ^α as follows: $\rho^\alpha(v)$ is the \mathbb{F} -vector space with basis $\{\beta \in [\alpha] \mid s\beta = v\}$ and $\rho^\alpha(e)$ is the linear transformation defined as

$$\beta\rho^\alpha(e) := \begin{cases} e_2e_3 \cdots & \text{if } \beta = ee_2e_3 \cdots \\ 0 & \text{otherwise.} \end{cases}$$

Grouping the paths starting at v in $[\alpha]$ by their first arrow gives Condition (I). Thus $[\alpha]$ is a basis for M^α and the proof of Theorem 3.2 shows that $\beta e^* = e\beta$ when $te = s\beta$ and 0 otherwise. Also $\beta pq^* = q\gamma$ if $\beta = p\gamma$ and 0 otherwise, implying that M^α is simple.

There are two types of M^α depending on whether α is (eventually) periodic (that is, we can find m, n so that $e_{k+n} = e_k$ for $k > m$) or not. When α is not periodic, M^α is a graded $L(\Gamma)$ -module: the degree of $f_1f_2f_3 \cdots$ is $m - n$ where m and n are the positive integers satisfying $f_{m+k} = e_{n+k}$

for all $k \in \mathbb{N}$. If α is periodic, picking the n, m above smallest possible with $C := e_{m+1}e_{m+2} \cdots e_{m+n}$, we get a bijection between the set of paths $P^C := \{p \mid tp = sC, p \text{ does not end with } C\}$ and $[\alpha]$ given by $p \leftrightarrow pC^\infty$. Via this identification the image of $pC^l q^*$ with $l \in \mathbb{N}$ and $p, q \in P^C$ in $\text{End}^{\mathbb{F}}(M^\alpha)$ is E_{pq} .

When Γ is R_n , the rose with n petals and $\alpha = e_0 e_0 e_0 \cdots$ then M^α above is (isomorphic to) the module of Example 4.1.

5 Support Subgraphs and The Monoid of a Finitely Separated Digraph

In this short section we collect a few definitions and facts needed in the later sections. We also reinterpret the criterion for the existence of a nonzero finite dimensional representation in terms of the nonstable K -theory of $L(\Gamma) := L_{\mathbb{F}}(\Gamma, \Pi)$.

- A subgraph $\Gamma' = (V', E')$ of $\Gamma = (V, E)$ is called **cohereditary** when for all e in E if $te \in V'$ then se also is in V' .
- A subgraph Γ' of the separated digraph Γ is **colorful** for any X in Π if $sX \in V'$ then $X \cap E' \neq \emptyset$.
- If M is a right $L(\Gamma)$ -module then the **support subgraph** of M , denoted by Γ_M , is the induced subgraph of Γ on $V_M := \{v \in V \mid Mv \neq 0\}$.

The subgraph $\Gamma' = (V', E')$ is cohereditary if and only if $V \setminus V'$ is a hereditary subset of V . When Γ' is full then Γ' is colorful if and only if $V \setminus V'$ is Π -saturated as defined in [11]. Our focus is more on the support subgraph rather than the ideal I_M generated by $V \setminus V_M = \{v \in V \mid Mv = 0\}$, so we work with cohereditary and colorful instead of hereditary and Π -saturated.

Lemma 5.1 *The following are equivalent for a subgraph Λ of a (finitely separated) digraph Γ :*

- (i) $\Lambda = \Gamma_M$, the support subgraph of a unital $L(\Gamma)$ -module M .
- (ii) Λ is a full, cohereditary and colorful subgraph.
- (iii) If $\Lambda = (V', E')$ then

$$\theta(v) = \begin{cases} v & v \in V' \\ 0 & v \notin V' \end{cases} \quad ; \quad \theta(e) = \begin{cases} e & e \in E' \\ 0 & e \notin E' \end{cases} \quad ; \quad \theta(e^*) = \begin{cases} e^* & e \in E' \\ 0 & e \notin E' \end{cases}$$

defines an onto algebra homomorphism $\theta : L(\Gamma) \longrightarrow L(\Lambda)$.

Proof. (i) \Rightarrow (ii): Γ_M is a induced subgraph hence full. If te is in Γ_M then $0 \neq Mte = Me^*e \subseteq Me = M(se)e$, so $Mse \neq 0$, that is, se is also in Γ_M and Γ_M is cohereditary. If sX is in Γ_M then $0 \neq MsX \cong \sum_{e \in X} Mte$ implies that

there is an e in X with $Mte \neq 0$. Thus e is in Γ_M and Γ_M is colorful.

(ii) \Rightarrow (iii): We need to check that θ preserves the defining relations of $L(\Gamma)$. No hypothesis is necessary to see that the path algebra relations are satisfied. For e, f in X if $e \neq f$ then $e^*f = 0$ in $L(\Lambda)$ as well as in $L(\Gamma)$. To see that $e^*e = te$ is preserved we need Λ to be cohereditary and full. (If $e \in E'$ then $te \in V'$ and $e^*e = te$ holds in $L(\Lambda)$ also. If $te \in V'$ then $se \in V'$ since Λ is cohereditary. So $e \in E'$ since Λ is full. Again $e^*e = te$ holds in $L(\Lambda)$. Otherwise $e^*e = 0 = te$ in $L(\Lambda)$.) Finally, if $sX \in V'$ then $X \cap E' \neq \emptyset$ since Λ is colorful. The image of $sX = \sum_{e \in X} ee^*$ under θ is $sX = \sum_{e \in E' \cap X} ee^*$, a defining relation of $L(\Lambda)$.

(iii) \Rightarrow (i): Given a subgraph $\Lambda = (V', E')$ so that $\theta : L(\Gamma) \rightarrow L(\Lambda)$ defines an algebra epimorphism, let $M := L(\Lambda) \cong L(\Gamma)/\text{Ker}\theta$. Now $v \in V'$ if and only if $\theta(v) \neq 0$ and $Mv = L(\Lambda)v \neq 0$. Hence the vertex set of Γ_M is V' . But Λ is full (if $te \in V'$ then $0 \neq \theta(te) = \theta(e^*)\theta(e)$, so $e \in E'$) hence $\Gamma_M = \Lambda$. ■

Proposition 5.2 *If M is a unital $L(\Gamma)$ -module then M also has the structure of a unital $L(\Gamma_M)$ -module (where Γ_M is the support subgraph of M) inducing the $L(\Gamma)$ structure via the epimorphism $\theta : L(\Gamma) \rightarrow L(\Gamma_M)$. Hence $\text{Ker}\theta \subseteq \text{Ann}M$.*

Proof. Let ρ_M be the quiver representation of Γ corresponding to M (as in Theorem 3.2). The restriction of ρ_M to Γ_M satisfies (I) because for all $X \in \Pi_M$

$$\rho_M|_{\Gamma_M}(sX) = MsX \xrightarrow{\cong} \bigoplus_{e \in X} Mte \cong \bigoplus_{e \in X \cap E_M} Mte = \bigoplus_{e \in X \cap E_M} \rho_M|_{\Gamma_M}(te)$$

(since $Mte = 0$ for $e \in X \setminus E_M$). Let M' be the unital $L(\Gamma_M)$ -module corresponding to $\rho_M|_{\Gamma_M}$. Now M' is also an $L(\Gamma)$ -module via $\theta : L(\Gamma) \rightarrow L(\Gamma_M)$. As vector spaces $M' = \bigoplus_{v \in V_M} Mv \cong \bigoplus_{v \in V} Mv = M$ by Lemma 3.1 and since $Mv = 0$ for $v \in V \setminus V_M$. We can define an $L(\Gamma_M)$ -module structure on M via this isomorphism. But the action of the generators $v \in V$, $e \in E$, $e^* \in E^*$ on M and M' is compatible with this isomorphism, so $M \cong M'$ as $L(\Gamma)$ -modules. Thus the $L(\Gamma)$ -module structure of M is induced from the $L(\Gamma_M)$ -module structure via θ . ■

As an $L(\Gamma_M)$ -module, M has full support, that is, $Mv \neq 0$ for all $v \in V_M$.

Remark 5.3 *If I_M is the kernel of $\theta : L(\Gamma) \rightarrow L(\Gamma_M)$ then I_M is generated by $V \setminus V_M = \{v \in V \mid Mv = 0\}$.*

Proof. Let J be the ideal generated by $V \setminus V_M$. Clearly $V \setminus V_M \subseteq I_M$, hence we have the projection from $L(\Gamma)/J$ to $L(\Gamma)/I_M \cong L(\Gamma_M)$. Conversely, let $\varphi : L(\Gamma_M) \rightarrow L(\Gamma)/J$ defined by: $\varphi(v) = v + J$, $\varphi(e) = e + J$, $\varphi(e^*) = e^* + J$. The defining relations of $L(\Gamma_M)$, except for (SCK2), are trivially satisfied. If

$X \cap E_M \neq \emptyset$ for $X \in \Pi$ then $\sum_{e \in X \cap E_M} ee^* + J = \sum_{e \in X} ee^* + J$ because $sX \in V_M$ so $e \in X \setminus E_M$ if and only if $te \notin V_M$. But φ is the inverse of the projection $L(\Gamma)/J \rightarrow L(\Gamma)/I_M$ thus $I_M = J$. ■

Definition 5.4 *The (additive) monoid $S(\Gamma)$ of the finitely separated digraph Γ is generated by V subject to the relations:*

$$sX = \sum_{e \in X} te \quad \text{for all } X \text{ in } \Pi.$$

Hence, dimension functions of Γ correspond exactly to monoid homomorphisms from $S(\Gamma)$ to \mathbb{N} (natural numbers under addition).

$S(\Gamma)$ is isomorphic to the monoid $\mathcal{V}(L(\Gamma))$ of nonstable K-Theory of $L(\Gamma)$, (that is, isomorphism classes of finitely generated projective $L(\Gamma)$ -module under direct sum). The generator v of $S(\Gamma)$ corresponds to the (right) projective $L(\Gamma)$ -module $vL(\Gamma)$ [12, Theorem 3.5], [11, Section 4], based on [15]. The corresponding relations among the isomorphism classes of the cyclic projective modules $vL(\Gamma)$ was shown to hold in the proof of Proposition 3.3.

6 Finite Dimensional Quotients of the Leavitt Path Algebra of a Row Finite Digraph

In this section, Γ will be a non-separated digraph, that is, $\Pi := \{s^{-1}(v) \mid s^{-1}(v) \text{ nonempty}\}$. In the non-separated context *finitely separated* means row-finite. Recall that an $L(\Gamma)$ -module is of *finite type* if $\dim^{\mathbb{F}}(Mv) < \infty$ for all $v \in V$. When V is finite, finite type is the same as finite dimensional since $M = \oplus_{v \in V} Mv$ by Lemma 3.1.

Lemma 6.1 *If an $L(\Gamma)$ -module M is finite type then the cycles of its support subgraph Γ_M have no exits.*

Proof. If $v_1, \dots, v_n, v_{n+1} = v_1$ are consecutive vertices in a cycle of Γ_M then $\dim(Mv_1) \geq \dim(Mv_2) \geq \dots \geq \dim(Mv_n) \geq \dim(Mv_1)$ by Lemma 3.1. Hence $\dim(Mv_k) = \dim(Mv_{k+1})$ for $k = 1, \dots, n$. It follows from Lemma 3.1 again that $Mte = 0$ for $e \in s^{-1}(v_k)$ unless $te = v_{k+1}$. Thus cycles of Γ_M have no exits. ■

Next we characterize all possible finite dimensional quotients of the Leavitt path algebra of a row-finite digraph as direct sums of matrix algebras over finite dimensional cyclic algebras.

Theorem 6.2 *If A is a finite dimensional quotient of the Leavitt path algebra $L(\Gamma)$ of a row-finite digraph Γ then $A \cong \bigoplus_{k=1}^m M_{n_k}(B_k)$, where each n_k is a positive*

integer, $B_k = \mathbb{F}[x]/(P_k(x))$ with $P_k(x)$ non-constant and $P_k(0) = 1$, $k = 1, 2, \dots, m$.

Proof. If $A = L(\Gamma)/I$ is a finite dimensional quotient of $L(\Gamma)$ then A is a unital $L(\Gamma)$ -module. Its support subgraph Γ_A is finite by Lemma 3.1 and the cycles of Γ_A have no exits by Lemma 6.1. Let I_A be the ideal generated by $V \setminus V_A = \{v \in V \mid L(\Gamma)v = Iv\} = V \cap I$ as in Remark 5.3. We have a homomorphism from $L(\Gamma_A) \cong L(\Gamma)/I_A$ onto $L(\Gamma)/I = A$ (since I_A is generated by $I \cap V$). So we may replace Γ with Γ_A , a finite digraph whose cycles have no exits.

$L(\Gamma_A)$ is isomorphic to a direct sum of matrix algebras over \mathbb{F} and/or $\mathbb{F}[x, x^{-1}]$ by [5, Theorem 3.8 and 3.10] (the number of summands of the form $M_n(\mathbb{F})$ is the number of sinks in Γ_A and the number of summands of the form $M_n(\mathbb{F}[x, x^{-1}])$ is the number of cycles in Γ_A) and $A \cong L(\Gamma_A)/J$. From now on we will identify $L(\Gamma_A)$ with this direct sum of matrix algebras.

Let π^k be the projection from $L(\Gamma_A)$ to the k th factor $M_{n_k}(\mathbb{F})$ or $M_{n_k}(\mathbb{F}[x, x^{-1}])$ and π_{ij}^k be π^k composed with the projection to the ij -th entry. Note that multiplying on the left by E_{li} and on the right by E_{jm} in the k -th coordinate moves the ij -th entry to the lm -th entry, hence $\pi_{ij}^k(J)$ is independent of ij (since J is an ideal). If $J_k := \pi_{ij}^k(J)$ then J_k is an ideal of \mathbb{F} or $\mathbb{F}[x, x^{-1}]$ and $J = \oplus M_{n_k}(J_k)$:

We have $J \subseteq \oplus M_{n_k}(J_k)$ by the definition of the J_k . To see the converse note that $\oplus M_{n_k}(J_k)$ is generated by $E_{ij}^k \alpha$ with $\alpha \in J_k$ where E_{ij}^k denotes the element with E_{ij} in the k -th coordinate and 0 all the others. If $\alpha = \pi_{ij}^k(\beta)$ with $\beta \in J$ then $E_{ii}^k \beta E_{jj}^k = E_{ij}^k \alpha$, thus $E_{ij}^k \alpha \in J$ and $\oplus M_{n_k}(J_k) \subseteq J$.

If $J_k \triangleleft \mathbb{F}$ then either $J_k = \mathbb{F}$, in which case the corresponding summand does not appear in A , or $J_k = 0$ and the summand $M_{n_k}(\mathbb{F}) \cong M_{n_k}(B_k)$ where $B_k = \mathbb{F}[x]/(x-1)$. If $J_k \triangleleft \mathbb{F}[x, x^{-1}]$ then either $J_k = \mathbb{F}[x, x^{-1}]$ so the corresponding summand does not appear in A , or $J_k = (P_k(x))$ and we may assume that P_k is non-constant, $P_k \in \mathbb{F}[x]$ and $P_k(0) = 1$ (multiplying with a power of x if necessary). Now $B_k := \mathbb{F}[x]/(P_k(x)) \cong \mathbb{F}[x, x^{-1}]/(P_k(x))$ and $A \cong \oplus M_{n_k}(\mathbb{F}[x, x^{-1}])/M_{n_k}(J_k) \cong \oplus M_{n_k}(B_k)$. ■

As mentioned in the final paragraph of section 2 above, the graded ideals of $L(\Gamma)$ when Γ is a row-finite digraph are in 1-1 correspondence with hereditary saturated subsets of vertices [12, Theorem 5.3]. This correspondence is given by sending a graded ideal I to $V \cap I$ and its inverse sends a hereditary saturated subset S of V to (S) , the (graded) ideal generated by S . In particular, if I is a graded ideal then $I = (I \cap V)$. A subset S of V is hereditary if and only if the induced subgraph on its complement $V \setminus S$ is cohereditary. Also S is saturated if and only if the induced subgraph on $V \setminus S$ is colorful for a non-separated digraph. Hence we can add a fourth equivalent condition to Lemma 5.1 (for Γ a row-finite digraph): There is a 1-1 correspondence between graded ideals $\{I\}$ of $L(\Gamma)$ and support subgraphs $\{\Gamma_M = (V_M, E_M)\}$ given by $I = (V \setminus V_M)$. Thus, for any ideal I of $L(\Gamma)$ the unique maximal graded ideal J contained in I is

$(V \setminus V_{L(\Gamma)/I})$. Moreover the modules $L(\Gamma)/I$ and $L(\Gamma)/J$ have the same support subgraph.

Corollary 6.3 *If I is a graded ideal of $L(\Gamma)$ with $\dim(L(\Gamma)/I)$ finite then $A := L(\Gamma)/I$ is isomorphic to a direct sum of matrix algebras over \mathbb{F} .*

Proof. When I is graded $I = (V \setminus V_A)$ as explained above. There are no cycles in Γ_A because $L(\Gamma_A) \cong L(\Gamma)/I$ is finite dimensional. Hence the only summands of $L(\Gamma)/I$ are matrix algebras over \mathbb{F} . ■

In order to state a necessary and sufficient criterion (in terms of the digraph Γ) for the existence of a nonzero finite dimensional quotient of $L(\Gamma)$ we need a few definitions. We say v *reaches* w , denoted $v \rightsquigarrow w$, if there is a (directed) path p in Γ such that $sp = v$ and $tp = w$. This defines a preorder (reflexive and transitive relation) on the vertices of Γ . If v and w are on a cycle then $v \rightsquigarrow w$ and $w \rightsquigarrow v$. Let U be the set of sinks and cycles of Γ . There is an induced preorder on U , also denoted by \rightsquigarrow . (This is a partial order on U if and only if the cycles of Γ are disjoint.) A sink or a cycle $u \in U$ is *maximal* if $u' \rightsquigarrow u$ only if $u' = u$.

The **predecessors** of v in V is $V_{\rightsquigarrow v} := \{w \in V \mid w \rightsquigarrow v\}$. If u and w are two vertices on a cycle C then they have the same predecessors, so $V_{\rightsquigarrow C}$ is well-defined. Let $\Gamma_{\rightsquigarrow v}$ be the induced subgraph on $V_{\rightsquigarrow v}$.

Theorem 6.4 *Let Γ be a row-finite digraph. $L(\Gamma)$ has a nonzero finite dimensional module (equivalently a nonzero finite dimensional quotient) if and only if Γ has a maximal sink or cycle with finitely many predecessors.*

Proof. Having a nonzero finite dimensional quotient is equivalent to having a nonzero finite dimensional module: Any quotient is also a module and conversely if M is a nonzero finite dimensional $L(\Gamma)$ -module then there is a nonzero homomorphism from $L(\Gamma)$ into $\text{End}(M)$ whose image is finite dimensional.

If there is a nonzero finite dimensional quotient $L(\Gamma)/I$ let $M = L(\Gamma)/I$ and $\Lambda := \Gamma_M$, its support subgraph. Λ is a finite digraph (Lemma 3.1) whose cycles have no exits (Lemma 6.1). If Λ has a sink w then w is also a sink in Γ because Λ is colorful by Lemma 5.1. There is no path from any cycle in Γ to w since this cycle would be a cycle with an exit in Λ (as Λ is cohereditary). Hence w is a maximal sink. If there is no sink then the finite digraph Λ must have a cycle. This cycle has to be maximal, as above, otherwise Λ would have a cycle with an exit. The predecessors of this maximal sink or cycle is contained in Λ so it is finite.

Conversely, if Γ has a maximal sink or cycle with finitely many predecessors then the induced subgraph Λ on this finite set W of predecessors is full, cohereditary and colorful. So $L(\Lambda)$ is a quotient of $L(\Gamma)$ by Lemma 5.1. Moreover, there is at most one cycle in Λ which has no exits. Thus $L(\Lambda) \cong M_n(\mathbb{F})$ if

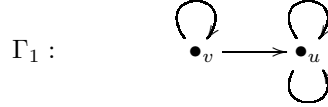
there is no cycle or $L(\Lambda) \cong M_n(\mathbb{F}[x, x^{-1}])$ when there is a cycle (as in the proof of Theorem 6.2 above). In both cases the finite dimensional algebra $M_n(\mathbb{F})$ can be realized as a quotient of $L(\Lambda)$ hence also of $L(\Gamma)$. ■

When Γ is a (non-separated) row-finite digraph all finite dimensional representations of $L(\Gamma)$ are classified in [21].

Corollary 6.5 *We have the following implications for the Leavitt path algebra of a finite digraph, neither of which is reversible: $L(\Gamma)$ has finite Gelfand-Krillov dimension implies that $L(\Gamma)$ has a nonzero finite dimensional quotient implies that $L(\Gamma)$ has IBN.*

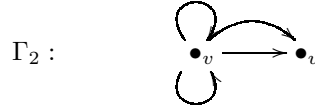
Proof. If $L(\Gamma)$ has finite Gelfand-Krillov dimension then the cycles in Γ are disjoint [8, Theorem 5]. Thus Γ must have a maximal sink or a maximal cycle and by Theorem 6.4, $L(\Gamma)$ has a nonzero finite dimensional quotient. If $L(\Gamma)$ has a nonzero finite dimensional quotient then $L(\Gamma)$ has IBN (since finite dimensional unital algebras have IBN and if a unital ring does not have IBN then neither does any nonzero homomorphic image of it).

The examples below show that neither implication is reversible:



There is no path to the loop at v from any other cycle, hence $L(\Gamma_1)$ has a nonzero finite dimensional quotient. But the Gelfand-Krillov dimension is infinite since the loops at u are not disjoint.

To see that the second implication is not reversible consider the digraph Γ_2 below:



$L(\Gamma_2)$ has no nonzero finite dimensional quotient (both cycles and the sink are reachable from another cycle). But $L(\Gamma_2)$ has IBN by the criterion of Kanuni-Özaydin [20]: The only relation we have is $v = 2v + 2u$, yielding $(1, 2)$. Then $L(\Gamma_2)$ has IBN since $(1, 1)$ is not in the \mathbb{Q} -span of $(1, 2)$. Another way to see that $L(\Gamma_2)$ has IBN is to note that $L(\Gamma_2)$ is isomorphic to a Cohn path algebra (of the rose with 2 petals). Cohn path algebras have IBN [6]. ■

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